BUTP-95/25

Perturbative solution of the Schwinger model

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Abstract

For the exactly solvable Schwinger model one interesting question is how to infer the exact solution from perturbation theory. We give a systematic procedure of deriving the exact solution from Feynman diagrams of arbitrary order for arbitrary n-point functions. As a byproduct, from perturbation theory we derive exact integral equations that the n-point functions have to obey.

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1 Introduction

The Schwinger model [1], which is QED_2 with one massless fermion, has devoted a long history of extensive study. The reason for this is that the Schwinger model combines two features that make it an ideal test labor for field theory studies and phenomenological methods. On one hand, it is exactly solvable and solutions within the operator formalism [2] or the path integral framework ([3], [4], [6], [7], [5]) are wellknown. On the other hand, it shares some nontrivial features with more realistic field theories: An anomaly is present and may be used for quantization of the fermion ([17], [11], [8], [18], [7]). Further instantons are present ([3], [4], [10], [6], [7]) and cause the formation of the fermion condensate and of a nontrivial vacuum (Θ-vacuum) ([2], [3], [4], [10], [6], [7], [5]). Besides, confinement of the fermion is realized in a precise manner: the fermionic n-point functions do not tend to free ones for large separations ([2], [12], [13], [14]).

One interesting question in this context is whether or how far it is possible to derive the exact solution and its nontrivial – partly nonperturbative – features from perturbation theory.

This question we investigate, and we find that by a systematic procedure we may transform all graphs of a given order into the corresponding contribution to the exact n-point function at hand. Further, from an investigation of Feynman graphs of arbitrary order we prove some exact integral equations to hold for specific n-point functions. These are related to the Dyson-Schwinger equations of the Schwinger model (see e.g. [5]).

The organization of the paper is as follows: first we review the exact path integral solution of the Schwinger model in Euclidean space and show how even contributions from nontrivial vacuum sectors may be derived from the trivial sector, being thereby accessible to perturbation theory.

Then we present the perturbative approach to the Schwinger model and derive the rules of how to reduce Feynman graphs to the exact solution. These computations turn out especially simple in a graphical fashion.

At last, from perturbation theory, we derive some exact integral equations for the n-point functions and comment on their meaning.

2 The exact path integral solution

As a starting point we use the exact vacuum functional after integration of the fermions (which can be found e.g. in [6], [5]),

$$Z[\lambda,\eta,\bar\eta] = \sum_{k=-\infty}^{\infty} Z_k[\lambda,\eta,\bar\eta],$$

$$Z_k[\lambda,\eta,\bar\eta] = N\int D\beta^{[k]} \prod_{i_0=0}^{k-1} (\bar\eta\Psi_{i_0}^{[\beta]})(\bar\Psi_{i_0}^{[\beta]}\eta)\cdot$$

$$\cdot\, e^{\int dxdy\bar\eta(x)G^{[\beta]}(x,y)\eta(y)} e^{\int dx(\frac{1}{2}\beta\mathbf{D}\beta+\frac{1}{e}\beta\lambda)} \tag{1}$$

where the gauge field is parametrized by the prepotential β

$$A_\mu = \frac{1}{e}\epsilon_{\mu\nu}\partial^\nu\beta \quad , \quad A_\mu J^\mu =: \beta\lambda$$

$$\Rightarrow \frac{\delta}{\delta J_\mu} = \epsilon^{\mu\nu}\partial_\nu \frac{\delta}{\delta \lambda} \tag{2}$$

corresponding to Lorentz gauge. $G^{[\beta]}$ is the exact fermion propagator

$$G^{[\beta]}(x,y) = e^{i(\beta(x)-\beta(y))\gamma_5} G_0(x-y),$$

$$G_0(z) = \frac{z^\mu \gamma_\mu}{2\pi(z^2 - i\epsilon)}. \tag{3}$$

$\mathbf{D}$ is the operator of the effective action of the prepotential β, with Green's function $\mathbf{G}$,

$$\mathbf{D} = \frac{1}{\pi\mu^2}\Box(\Box - \mu^2) \quad , \quad \mu^2 = \frac{e^2}{\pi}$$

$$\mathbf{G}(x) = \pi(D_\mu(x) - D_0(x))$$

$$\mathbf{D}_x \mathbf{G}(x-y) = \delta(x-y) \tag{4}$$

where μ is the "photon" mass and $D_\mu(x), D_0(x)$ are the massive and massless scalar propagators in two dimensions:

$$D_0(x) = \frac{1}{4\pi}\ln(x^2 - i\epsilon) + \text{const.} \quad , \quad D_\mu(x) = -\frac{1}{2\pi}K_0(\mu|x|), \tag{5}$$

$K_0(z)$ being the McDonald function (for details see e.g. [6], [7], [5]).

k is the instanton number and $\Psi_{i_0}^{[\beta]}$ are the zero modes corresponding to the gauge field β with instanton number k. (Their precise form can be found in [6], [7], [5]).

Due to the Gaussian nature of the path integral (1) and the knowledge of the exact fermion propagator in an external field (3) all n-point functions of the theory may be computed explicitly. E.g. for the fermionic two-point function we find (for the sectors $k = 0, \pm 1$; higher sectors do not contribute)

$$\langle T\bar{\Psi}^\beta(y)\Psi^\alpha(x)\rangle^{k=0} = G_0^{\alpha\beta}(y-x)e^{\mathbf{G}(0)-\mathbf{G}(x-y)} \tag{6}$$

$$\langle T\bar{\Psi}^\beta(y)\Psi^\alpha(x)\rangle^{k=\pm 1} = \frac{1}{2\pi}P_\pm^{\alpha\beta}e^{\mathbf{G}(0)+\mathbf{G}(x-y)}. \tag{7}$$

We observe a feature that persists to hold for higher n-point functions: for vector-like fermionic bilinears only the zero sector contributes, whereas for chiral bilinears the contributing instanton sector is fixed by chirality. Therefore for an n-point function consisting of an arbitrary number of vectors, n_+ positive chirality bilinears $(\bar{\Psi}(y)P_+\Psi(x))$ and n_- negative chirality bilinears only the sector $k = n_+ - n_-$ contributes.

General n-point functions always may be decomposed into a sum of n-point functions of definite chirality. In fact, this may be used to compute all n-point functions from the trivial vacuum sector via clustering. E.g. for the four-point function we find $(S_\pm(y,x) \equiv \bar{\Psi}(y)P_\pm\Psi(x))$

$$\langle TS_+(y_2,y_1)S_-(x_2,x_1)\rangle \equiv \langle T\bar{\Psi}(y_2)P_+\Psi(y_1)\bar{\Psi}(x_2)P_-\Psi(x_1)\rangle^{k=0} =$$

$$\frac{(x_1-y_2)^\mu}{2\pi(x_1-y_2)^2}\frac{(x_2-y_1)_\mu}{2\pi(x_2-y_1)^2}\cdot e^{2\mathbf{G}(0)+\mathbf{G}(x_1-x_2)+\mathbf{G}(y_1-y_2)-\sum_{i,j}\mathbf{G}(x_i-y_j)}. \tag{8}$$

Performing now the limit $y_j \to y_j + a$, $a \to \infty$, $y_1 - y_2$ fixed, we find that the free fermion propagators just cancel against the large range part of $\exp\sum \mathbf{G}(x_i - y_j)$ (the massless propagator

D_0; the massive one, D_μ, vanishes exponentially). Comparing with the expression (7) for the two-point function we therefore find

$$\lim_{a\to\infty}\langle TS_+(y_2+a,y_1+a)S_-(x_2,x_1)\rangle^{k=0} = \langle TS_+(y_2,y_1)\rangle^{k=1}\langle TS_-(x_2,x_1)\rangle^{k=-1} \tag{9}$$

which is the cluster property.

Generally, any VEV can be computed from the trivial sector by multiplying the operators of given chiralities by their opposite chirality counterparts and by the use of the cluster property.

3 Schwinger model perturbation theory

In our perturbative calculations we will always use the exact photon propagator, which consists of a sum of all possible vacuum polarization insertions,

and all other possible contributions vanish in the Schwinger model (for details see e.g. [16], [15], [8], [7]). In momentum space it is

$$D^{\mu\nu}(p) = (g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})\frac{1}{p^2+\mu^2} = -\epsilon^{\mu\lambda}p_\lambda\epsilon^{\nu\rho}p_\rho\frac{1}{p^2(p^2+\mu^2)} \tag{10}$$

as can be easily computed from the vacuum functional (1) or within perturbation theory ([8]). The propagator looks like being in "Landau gauge", however this is the unique and gauge invariant result. When inserted into a Feynman diagram, this propagator gives rise to

$$-e\gamma_\mu\epsilon^{\mu\lambda}p_\lambda\ldots\frac{1}{p^2(p^2+\mu^2)}\ldots e\gamma_\nu\epsilon^{\lambda\rho}p_\rho = -\not{p}\gamma_5\ldots\not{p}\gamma_5\tilde{\mathbf{G}}(p^2), \tag{11}$$

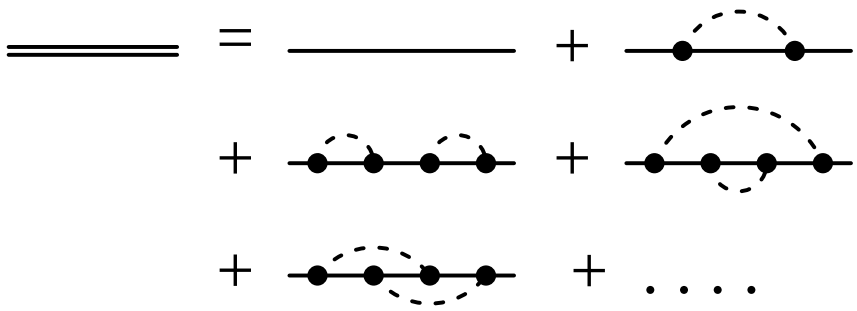
$\tilde{\mathbf{G}}(p^2)$ being the Fourier transform of $\mathbf{G}(x)$, eq. (4).

In actual computations, when an exact photon line begins and ends at the same fermion line, there is always an even number of γ matrices between the two γ_5, therefore we may omit them in the computations (in more general cases we will reinsert them when it is necessary).

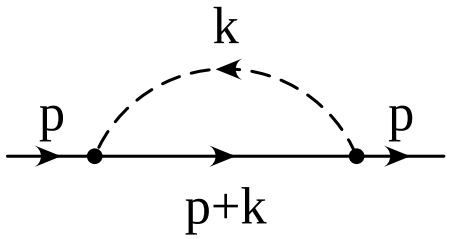
Further we will frequently use the identity

$$\frac{1}{\not{p}}\not{k}\frac{1}{\not{p}+\not{k}} = \frac{1}{\not{p}} - \frac{1}{\not{p}+\not{k}}. \tag{12}$$

Let us take the fermionic two-point function as a first example. Up to second order in the exact photon propagator we have



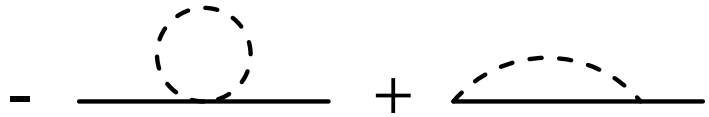
For the first order graph



we obtain, using (11) and (12)

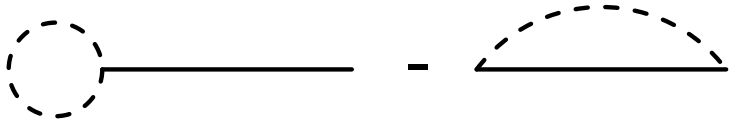
$$-\int \frac{dk}{(2\pi)^2}\mathbf{G}(k)\frac{1}{\not p}\not k\frac{1}{\not p+\not k}\not k\frac{1}{\not p} =$$

$$-\int \frac{dk}{(2\pi)^2}\mathbf{G}(k)\frac{1}{\not p}\not k\frac{1}{\not p} + \int \frac{dk}{(2\pi)^2}\mathbf{G}(k)\frac{1}{\not p+\not k}\not k\frac{1}{\not p} = \tag{13}$$



and the first term vanishes. The final result is

$$\int \frac{dk}{(2\pi)^2}\mathbf{G}(k)\frac{1}{\not p} - \int \frac{dk}{(2\pi)^2}\mathbf{G}(k)\frac{1}{\not p+\not k} = \tag{14}$$

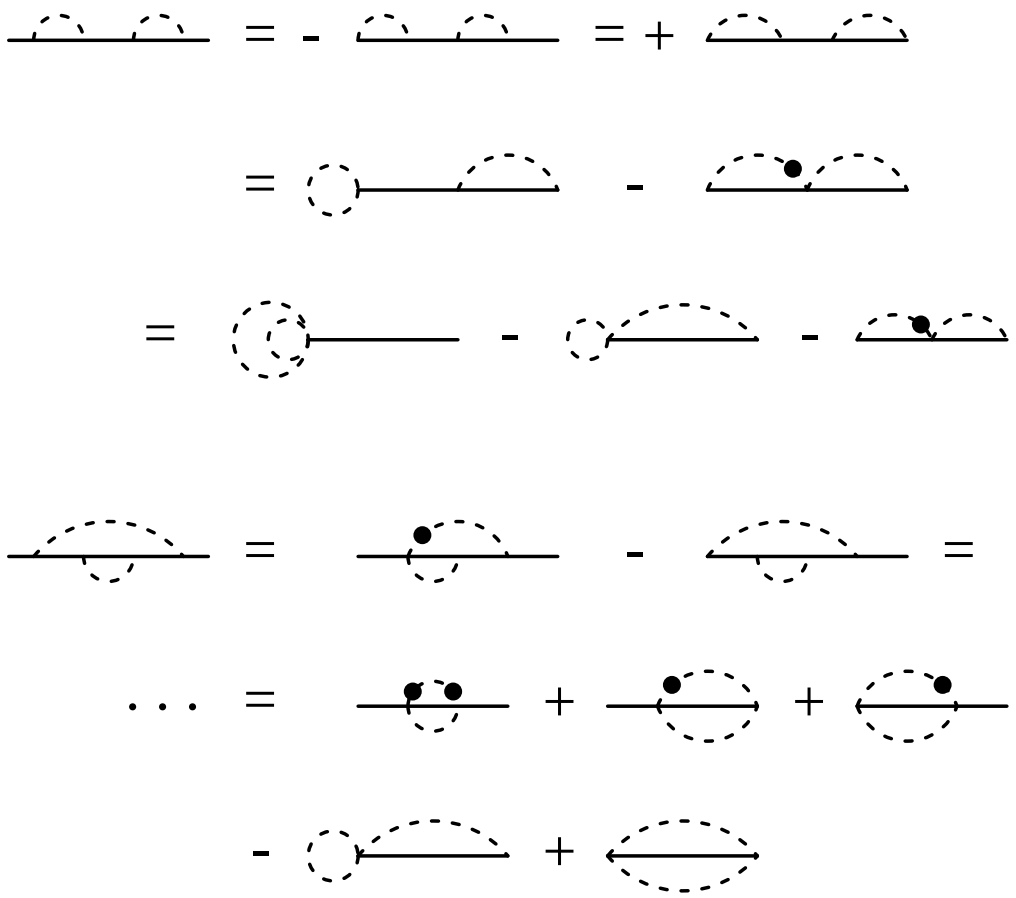


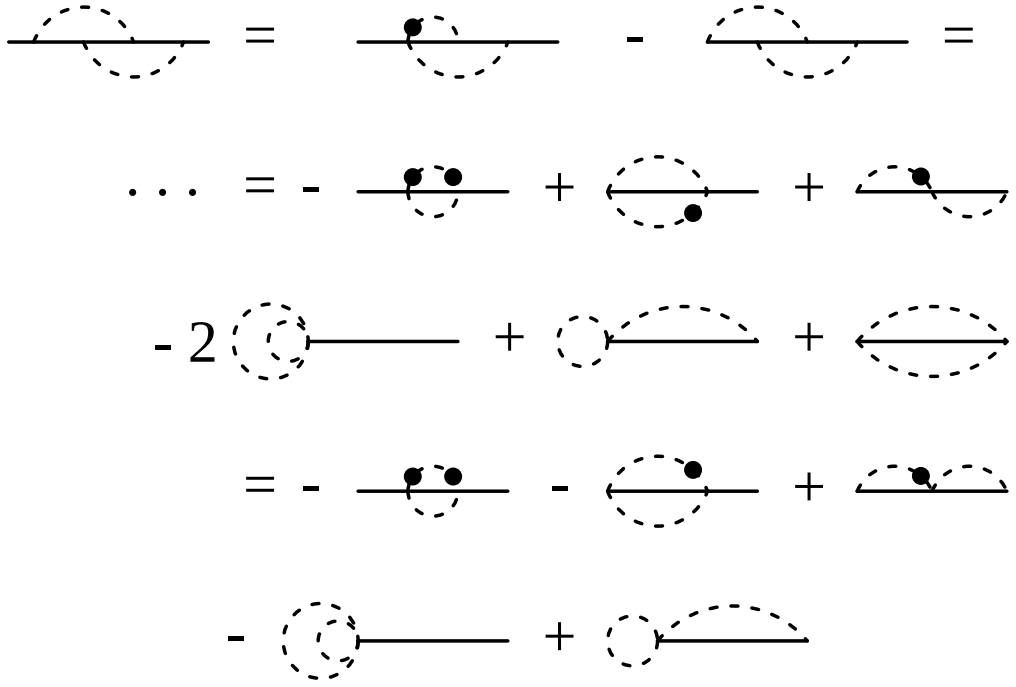
which, after Fourier transformation, obviously is the first order (in $\mathbf{G}$) expansion of the exact solution (6).

From this simple example we derive the following graphical rules: starting at any vertex, you obtain two graphs by leaving out the inner or outer fermion propagator (the latter one with a minus sign). Photon lines get a dot at that vertex that has been changed (however photon lines running to the very begin or end always have been changed, so it is not necessary there). Closed loops vanish when only one of the two vertices has been changed. Otherwise they decouple from all momenta and may be drawn to the begin or end of the fermion line. Whenever a changed vertex coincides with an unchanged one, the unchanged one cannot be removed further.

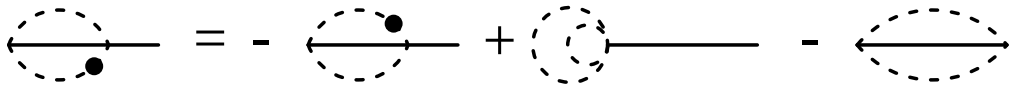
By applying these rules, we may reduce each individual graph to a final form. Then we have to sum all graphs of a given order, and only the sum will lead to the exact solution.

Applying these rules to the second order graphs we find



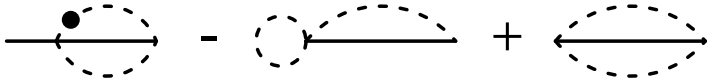


where we used

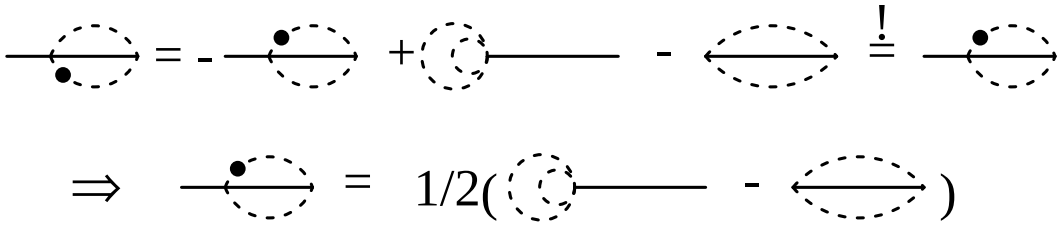


$$\int d\mu(k,l)\frac{1}{\not p+\not k+\not l}\not k\frac{1}{\not p} = -\int d\mu(k,l)\frac{1}{\not p+\not k+\not l}\not l\frac{1}{\not p}+$$
$$+\int d\mu(k,l)[\frac{1}{\not p}-\frac{1}{\not p+\not k+\not l}] \qquad (15)$$

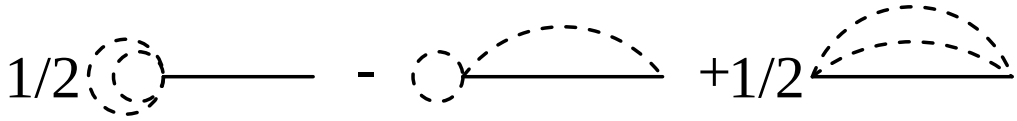
in the last step (the symmetric integration measure $d\mu(k,l)$ we display in a moment). For the sum of all three graph we get



In a next step we have to use the symmetry with respect to the two momentum integrations,



to obtain the final result



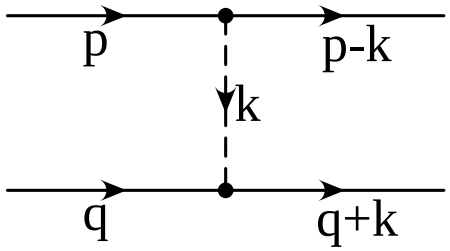
$$\int d\mu(k,l)[\frac{1}{2}\frac{1}{\not p} - \frac{1}{\not p + \not k} + frac12\frac{1}{\not p + \not k + \not l}], \tag{16}$$

$$d\mu(k,l) = \frac{dk}{(2\pi)^2}\frac{dl}{(2\pi)^2}\tilde{\mathbf{G}}(k)\tilde{\mathbf{G}}(l).$$

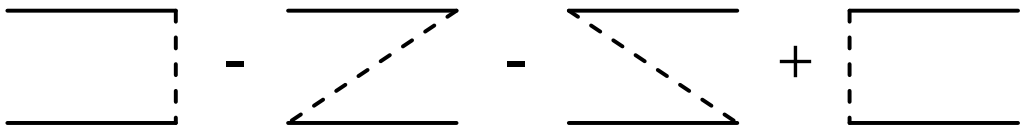
This result, after a Fourier transformation, obviously is the second order (in $\mathbf{G}$) of the exact solution (6).

So the general way of obtaining the exact solution is: reduce all graphs of a given order according to the graphical rules, rearrange some dotted graphs (like in (15)) in order to cancel them, and when some dotted graphs are left, rewrite them by using symmetric momentum integration. This recipe remains the same for higher n-point functions. Only when translating back into formulae you have to insert one γ_5 matrix at the beginning of each fermion line that carries an *odd* number of vertices.

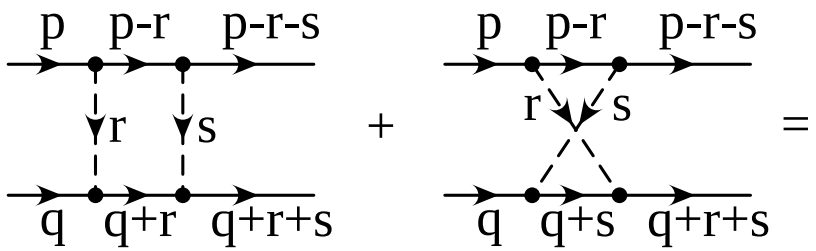
E.g. for the exchange part of the four-point function

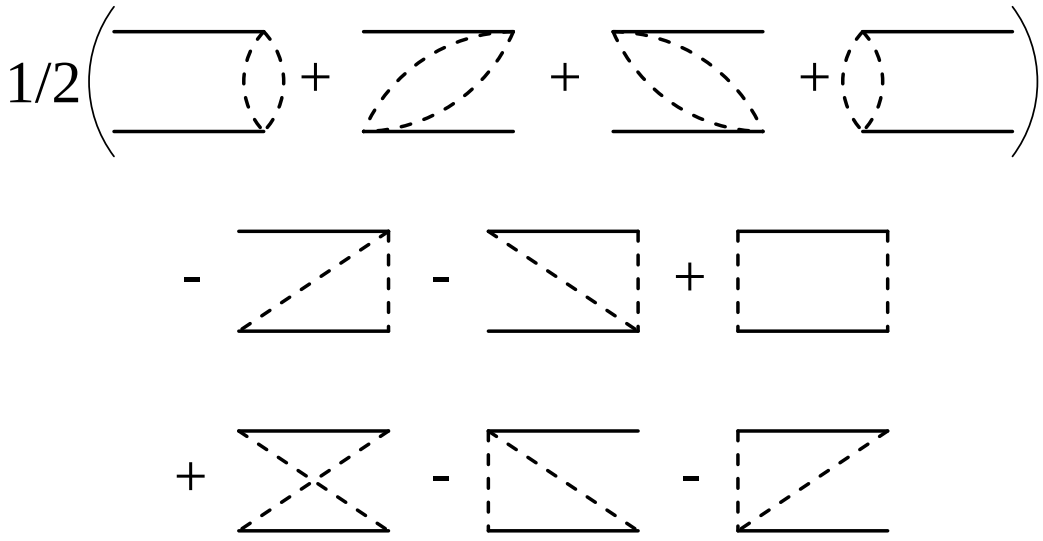


one obtains



for the first order and



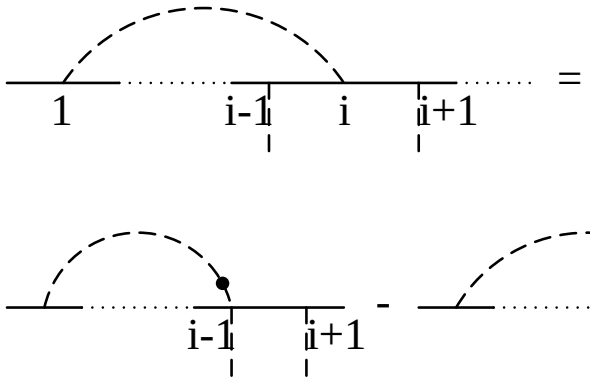


in second order. These results coincide with the first and second order expansions of the exact result (8).

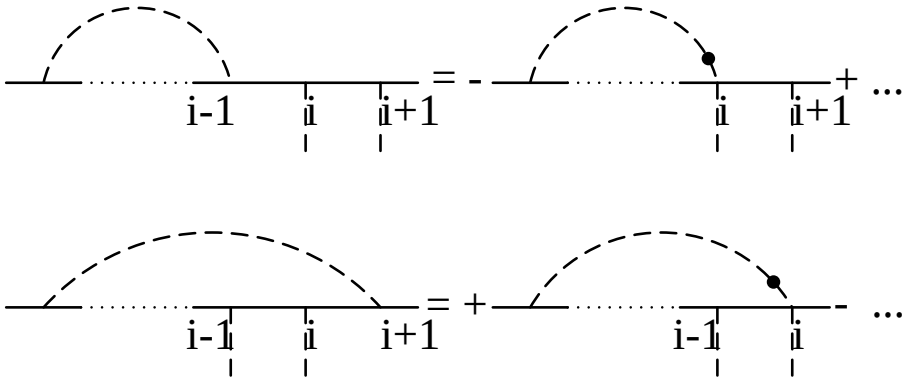
So we found systematic rules to derive the exact solution order by order, but the question remains if there is a way to derive the whole exact solution at once from perturbation theory; and indeed there is. More precisely, for any n-point function we may derive an exact integral equation from perturbation theory. This we discuss in the next section.

4 Integral equations for n-point functions

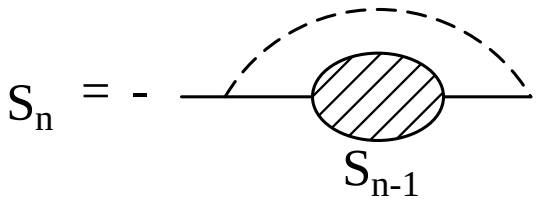
Again, we start with the two-point function (for the two-point function an analogous computation in light cone gauge has been done in [19]). Let us look at a graph of arbitrary order where the first photon runs from vertex 1 to some vertex i:



whereas when running from 1 to $(i-1), (i+1)$ (all other lines being unchanged) we find



and the diagram with $1 \dots i$ is cancelled by contributions from $1 \dots (i-1)$ and $1 \dots (i+1)$ (the diagrams that cancel are indeed identical although we indexed the vertices differently). Now, summing all graphs of a given order, for an arbitrary $1 \dots i$ diagram there exist the corresponding $1 \dots (i-1)$ and $1 \dots (i+1)$ diagrams, therefore nearly all diagrams cancel against each other. The exceptions are the $1 \dots 2$ diagram, when vertex 2 is removed towards vertex 1 (this however leads to a closed loop that is zero according to our rules), and the $1 \dots 2n$ diagram (for diagrams of order n). Here the vertex $2n$ has to be removed towards the very end of the graph. Because this cancellation argument is true for arbitrary distributions of the remaining photon lines, we get a recursion formula for the two-point function of nth order:



$$S_n(p) = -\int \frac{dk}{(2\pi)^2} \tilde{\mathbf{G}}(k) \frac{1}{\not{p}} \not{k} S_{n-1}(p-k)$$

$$\not{p}S_n(p) = -\int \frac{dk}{(2\pi)^2}\tilde{\mathbf{G}}(k)\not{k}S_{n-1}(p-k) \tag{17}$$

leading to the integral equation (with the zeroth order as inhomogenous part)

$$\not{p}S(p) = 1 - \int \frac{dk}{(2\pi)^2}\tilde{\mathbf{G}}(k)\not{k}S(p-k). \tag{18}$$

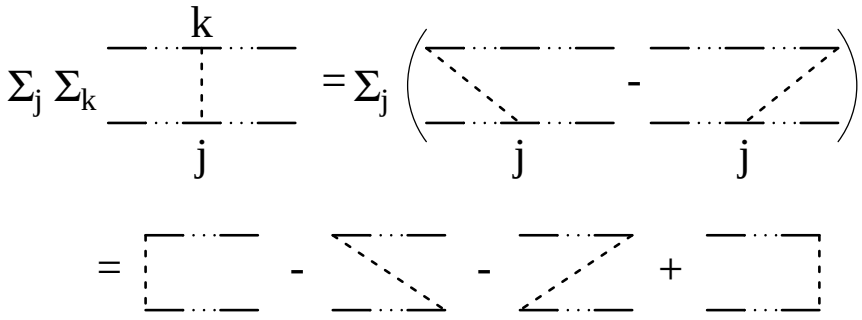
It is very easy to see that the exact solution (6) solves this equation:

$$S(x) = G_0(x)e^{\mathbf{G}(0)-\mathbf{G}(x)}$$

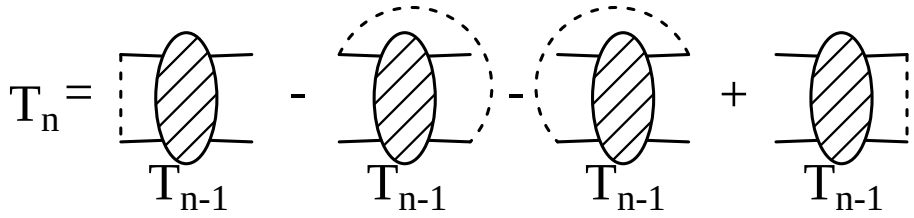
$$\partial\!\!\!/ S(x) = i\delta(x) - (\partial\!\!\!/\mathbf{G}(x))S(x) \tag{19}$$

which is the Fourier transform of the above equation. In fact, eq. (18) is just the momentum space version of the Dyson-Schwinger equation for the two-point function (for a general discussion of the Dyson-Schwinger equations for the Schwinger model in x space see [5]; there the discussion is for all instanton sectors).

The integral equation for the four-point function may be derived in a similar fashion. Arguing in the same way like before we may derive



or as a recursion formula



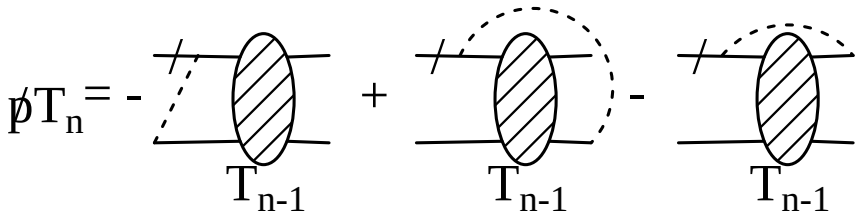
$$T_n^{\alpha\beta\gamma\delta}(p,q,p-k,q+k) = \int \frac{dk_2}{(2\pi)^2}dk_1\delta(k-k_1-k_2)\tilde{\mathbf{G}}(k_2)\gamma_5^{\alpha\alpha'}\gamma_5^{\beta\beta'}\cdot$$

$$\cdot[T_{n-1}^{\alpha'\beta'\gamma\delta}(p-k_2,q+k_2,p-k,q+k) - T_{n-1}^{\alpha'\beta'\gamma\delta}(p-k_2,q,p-k,q+k_1) -$$

$$T_{n-1}^{\alpha'\beta'\gamma\delta}(p,q+k_2,p-k_1,q+k) + T_{n-1}^{\alpha'\beta'\gamma\delta}(p,q,p-k_1,q+k_1)] \tag{20}$$

where however this operation only rises the number of *exchanging* photons. The "self energy" part is, of course, computed like for the two-point function, eq. (17).

In order to derive an equation analogous to (18), we again multiply by $\not{p}$, amputating thereby the upper incoming fermion line. As a consequence, the first upper vertex is fixed, and we obtain (including now the self energy):



(the slash indicates amputation) and therefore the equation (in symbolic notation)

$$\not{p}T(p,q,p-k,q+k) = \mathbf{1}\otimes S(q)\delta(k)+$$

$$(\gamma_\mu\gamma_5\otimes\gamma_5)\times\int\frac{dk_2}{(2\pi)^2}dk_1\delta(k-k_1-k_2)k_2^\mu\tilde{\mathbf{G}}(k_2)\cdot$$

$$\cdot[-T(p-k_2,q+k_2,p-k,q+k)+T(p-k_2,q,p-k,q+k_1)]-$$

$$(\gamma_\mu\otimes\mathbf{1})\times\int\frac{dk_2}{(2\pi)^2}dk_1\delta(k_1-k-k_2)k_2^\mu\tilde{\mathbf{G}}(k_2)T(p-k_2,q,p-k_1,q+k) \qquad (21)$$

(for the inhomogenous part we used our knowledge of the two-point function). This is the momentum space version of the Dyson-Schwinger equation for the four-point function, see [5].

The generalization to higher n-point functions is straight forward.

5 Summary

We have given a systematic procedure of finding the exact n-point functions from a resummed perturbation theory order by order. We have explicitly demonstrated the procedure for the fermionic two- and four-point functions, but generalizations are straight forward. Further we have derived integral equations for the exact n-point functions from perturbative recursion relations. By solving these equations the n-point functions in principle can be found, which corresponds to a complete summation of the perturbation series. By the way, these equations are just the Dyson-Schwinger equations (in momentum space) which the exact n-point functions have to obey.

By these methods, and having in mind that the trivial instanton sector suffices to obtain the complete solution of the Schwinger model, all features of the Schwinger model in principle can be treated by perturbative methods.

Acknowledgements

The author thanks H. Leutwyler and A. V. Smilga for very helpful discussions and the members of the Institute of Theoretical Physics at Bern University, where this work was done, for their hospitality.

This work was supported by a research stipendium of the University of Vienna.

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